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*Publication date:*  
1996

[Link to publication in Tilburg University Research Portal](#)

*Citation for published version (APA):*

Kort, P. M. (1996). *Optimal R&D Investments of the Firm*. (CentER Discussion Paper; Vol. 1996-47). Operations research.

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# Optimal R&D investments of the firm

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## Abstract

This paper examines irreversible decisions on innovative activities where it takes time to complete an R&D project. The total amount of R&D investments that the firm needs to undertake in order to obtain the breakthrough in the innovation process is uncertain. R&D investments are limited by the restriction that they must be self-financed.

It is shown that R&D investments are more valuable when the level of uncertainty is large. Especially, it is very attractive to undertake R&D investments if a project faces many uncertainties during its early phases. Furthermore we study how R&D behavior is influenced by different levels of the discount rate and the financing limit. Moreover, the effects of R&D subsidies, spillover benefits and a payoff that decreases over time are analyzed.

JEL classification: 522, 621.

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<sup>immediate</sup>The author thanks Henk Norde for computational assistance and Hossein Farzin, Talitha Feenstra, Steffen Jørgensen and Piet Verheyen for their constructive comments.

# 1 Introduction

Recent years have witnessed the development of a large literature analyzing the incentives for firms to perform innovative activities. Besides the large stream of macroeconomic publications on the so-called "endogenous growth theory" (see, e.g., Romer (1990a, 1990b), Grossman and Helpman (1990, 1991)), there exists a considerable amount of microeconomic contributions that can be subdivided in strategic and decision-theoretic literature. The strategic literature aims at innovating behavior of firms while taking into account the reactions of competitors. This stream of literature focusses on aspects of rivalry, thus game-theoretic approaches (see, e.g., Reinganum (1985) and Tirole (1988)).

Within the decision-theoretic literature on innovation, dynamic R&D investments are treated as an optimal control problem facing a single firm. This kind of analysis applies most directly to situations in which a monopolist undertakes a research project, and an entrepreneur pursues a new patent without rivalry, and a competitive firm seeks an improved technology. Of particular importance in the decision-theoretic framework is the assumption that the firm believes its choice of the level of R&D spending does not influence the R&D spending of its rivals. This assumption may be justified when there are perhaps many ways of achieving the same innovation, so that the firm's rivals may not be in its present line of business (Kamien and Schwartz (1982), p. 107). Recent contributions within this area are Grossman and Shapiro (1986), Majd and Pindyck (1987) and Gifford (1992).

Besides performing R&D itself the firm has the possibility of technology transfer, but also then it needs to have substantial inhouse capacity in order to recognize, evaluate, negotiate and finally adopt the technology potentially available from others. Moreover, inhouse R&D has the additional advantage of facilitating better information flow from the R&D laboratory to those who would have to implement the new technology, and from the latter to the former. Therefore R&D growth has not led to a comparable process of market-based diffusion of labor and the emergence of specialized "innovation suppliers". Firms make innovations largely on the basis of inhouse technology, but with some contributions from other firms, and from public knowledge (Dosi (1988), p. 1130).

The purpose of this paper is to study optimal dynamic R&D behavior of a firm within a decision theoretic framework. Completing the project requires investing in R&D. The

actual total of R&D investments that needs to be undertaken to complete the project is negatively correlated with the present and future progress made by the firm's R&D laboratory. Since this progress is hard to predict, the cost to obtain the innovation is in general unknown. When after a while it turns out to be expensive to complete the project, the firm can stop investing in it.

We conclude that an R&D project takes time to complete and is subject to cost uncertainty. Pindyck (1993) distinguishes two types of cost uncertainty. The first is technical uncertainty which relates to the physical difficulty of completing a project: how much time, effort and materials will ultimately be required. The second kind of uncertainty relates to input costs, and is external to what the firm does. It arises when the prices of labor, land, and materials needed to build a project fluctuate considerably.

The cost uncertainty in completing an R&D project is mainly of the technical type. Therefore, since in this contribution we explicitly want to focus on R&D investments, we will neglect input cost uncertainty. As Pindyck (1993) argues, technical uncertainty can only be resolved by undertaking the project; actual costs and construction time unfold as the project proceeds. These costs may be greater or less than anticipated if impediments arise or if the work progresses faster than planned, but the total cost of investment is only known for certain when the project is completed. In the model specification of Pindyck (1993), technical uncertainty is the same for each phase of the project. However, in the beginning of an R&D project one is usually very uncertain about how much time, effort and materials will be needed to complete the project; in fact in some cases it is not even known whether the project can be realized at all. This implies that technical uncertainty is very large when the project is far from being completed. This uncertainty will reduce when some research has been undertaken, or, in other words, investing time and effort in the project reveals some information about what exactly is needed to develop the innovation. Hence, the level of technical uncertainty is larger in early phases of an R&D project. Therefore, in this paper we modify the model of Pindyck (1993), where technical uncertainty is the same for each phase of the project, in order to include the feature that working on the project reduces the project's uncertainty.

In the model R&D investments are limited by the restriction that they must be self-financed. According to Kamien and Schwartz (1982) this is realistic for the following two reasons. First, the uncertainty associated with innovative activities is strong, and

borrowing for risky projects is prohibitively expensive (Geroski (1992)) since the outcome of a risky project is uncertain and when it fails it leaves behind few tangible assets. Second, the firm may be reluctant to reveal detailed information about the project that would make it attractive to outside lenders, fearing its disclosure to potential rivals (Kamien and Schwartz (1982), p. 128). These observations are supported by anecdotal evidence and an empirical study of Guerard, Bean and Andrews (1987) who found out that there is a lack of statistical significance of R&D and external funds.

Besides determining the effects of the amount of uncertainty, the division of uncertainty across the different phases of the project, the level of the discount rate and the financing restriction on R&D investments, we also analyze the effects of three characteristics that are typical for innovation projects. First, we study the effect of an investment grant. This grant is donated to the firm by the government in order to stimulate innovative activities.

Second, we analyze what happens when spillover benefits exist. Existence of spillover benefits captures the fact that knowledge is a nonrival input (Romer (1990a)) in the sense that several firms may put the same information to use simultaneously at no extra cost. Here, the firm takes advantage of the more general forms of scientific and engineering knowledge generated outside the firm by competitors or universities.

Third, we analyze the effect that a decreasing value of a completed innovation over time can have on the firm's innovative activities. This value is equal to the payoff that accrues to the firm after the innovation project is completed. This payoff can decrease over time, because, first, the sooner the firm perfects the device, the longer is the time period that it captures monopoly rents. Second, the number of competitors that already has achieved the breakthrough in the innovation process will increase over time, implying that competition in the market of the innovated product also increases over time.

In Section 2 we develop the model, while the optimal investment rule is derived in Section 3. Here we also study how R&D investments depend on the amount of uncertainty, the division of uncertainty across the phases of the project, the level of the discount rate, and the availability of funds. Section 4 treats the effects of an R&D investment grant, spillover benefits and a decreasing payoff. The paper is concluded in Section 5.

## 2 The Basic Model

The problem of determining optimal R&D investments over time is a sequential investment problem. An R&D project takes time to complete, because during some time investments in machines, materials, human capital, etc., are needed in order to develop the innovation. The project generates no cash flow as long as it is not completed.

An R&D project is subject to technical uncertainty since it is not known beforehand how much time, effort and materials will ultimately be required to complete the project. However, every time the firm invests, some information about these future costs is revealed. When, based on this new information, the firm concludes that it is too expensive to complete the project, it can be stopped midstream.

Denote the actual total of R&D investments needed to complete the project at time  $t$  by  $\tilde{K}(t)$ . Due to the technical uncertainty  $\tilde{K}$  is a random variable that cannot be observed at time  $t$ . What is known at this time is the firm's expectation about how many R&D investments are needed to obtain the innovation, which is equal to  $K(t) = E(\tilde{K}(t))$ . In order to make things not too complicated, the value of a successful innovation,  $P$ , is assumed to be constant and known with certainty.

In Pindyck (1993) a general sequential investment problem (so, not only R&D but also, e.g., large construction works) is formulated subject to input cost uncertainty and technical uncertainty. He modelled technical uncertainty in the following way:

$$dK(t) = -I(t)dt + \gamma(I(t))^{\frac{1}{2}}(K(t))^{\frac{1}{2}}dw(t), \quad (1)$$

where  $I(t)$  is the investment at time  $t$ ,  $\gamma$  is a parameter that is constant over time and  $dw(t)$  is the increment of a Wiener process.

The project manager estimates the expected cost to completion, but due to the stochastic term this value will be adjusted over time. This adjustment is based on activities in developing the project. Therefore,  $K$  can only change when the firm undertakes some investments. This is the case as long as the firm does not invest in the project no additional information about this project becomes available, so that nothing changes about the firm's expectation concerning future costs. Hence, here there is no value of

waiting”.<sup>1</sup>

When the firm invests, the expected change in  $K$  over an interval  $\Delta t$  equals  $-I\Delta t$ , but the realized change can be greater or less than this, and  $K$  can even increase. As the project proceeds, progress will at times be slower and at times faster than expected. The variance of  $\tilde{K}$  falls as  $K$  falls, but the actual total amount of investments is only known when the project is completed.

From equation (1) we derive that

$$\text{var} \left( \frac{dK}{K} \right) = \gamma^2 \frac{I}{K} dt. \quad (2)$$

Hence, the variance of  $dK/K$  increases linearly with  $I/K$ . This means that, if for different values of  $K$  a fixed part of the expected total of investments is undertaken (i.e.  $I/K$  is constant), then the uncertainty concerning the part that the expected total of investment is reduced is constant too. Therefore, under equation (1) it holds that technical uncertainty is the same for all phases of the project.

However, for an R&D project it is in general the case that uncertainty is much larger during early phases of the project; at the start it is usually not known at all how much time, effort and materials will ultimately be required to develop the innovation, while in the end it is more or less clear what is left to be done. This implies that technical uncertainty is larger when  $K$  is large than when  $K$  is low. Hence, to make expression (1) suitable for modelling R&D investments, we must modify such that the variance of  $dK/K$  is increasing in  $K$  if a fixed part of the expected total of investment is undertaken. Therefore, we replace expression (1) by

$$dK(t) = -I(t)dt + \gamma(K(t))^\delta (I(t))^{\frac{1}{2}} (K(t))^{\frac{1}{2}} dw(t), \quad (3)$$

where  $\delta$  is a positive constant. We conclude that now technical uncertainty is much larger during early phases of the project (i.e.  $K$  is large) if  $\delta$  is large.

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<sup>1</sup>In case there is a positive value of waiting it is implied that undertaking investments now introduces opportunity costs that are ignored in the classical net present value (NPV) rule. This failure of the NPV criterion forms the key message of a stream of recent contributions in which it is argued that, due to this positive option value of postponing investment, a project with a positive NPV might still be uneconomical (see e.g. Dixit and Pindyck (1994)).

For reasons already stated in the Introduction, R&D investments can only be financed by retained earnings. If we define  $R$  ( $R > 0$  and constant) as the firm's profit flow before the breakthrough in the innovation process,  $I(t)$  is restricted from above by  $R$ . Knowledge acquisition is also irreversible, so the R&D investment at time  $t$  faces the constraint

$$0 \leq I(t) \leq R. \quad (4)$$

If we define the discount rate to be  $r$  ( $r > 0$  and constant), the problem is to find a stream of R&D outlays that maximizes the value of the innovation opportunity given by

$$F(K(t)) = \max_{I(t)} E_0 \left[ P e^{-r\tilde{T}} - \int_0^{\tilde{T}} I(t) e^{-rt} dt \right], \quad (5)$$

subject to expressions (3), (4) and  $K(\tilde{T}) = 0$ .

Uncertainty is related to the progress made in completing the R&D project. This kind of uncertainty results from the inability to predict how difficult a project will be, and this has nothing to do with the state of the overall economy. Therefore, in our model uncertainty is largely diversifiable, so that the discount rate equals the riskless rate.

In the next section we solve the basic model and study how sensitive R&D investments are with respect to the division of uncertainty across different phases of the project, the overall level of uncertainty, the discount rate and the availability of funds. Moreover, in Section 4 we extend this model by including some characteristics that are typical for an R&D investment problem, such as investment subsidies, spillover benefits and a decreasing payoff over time.

### 3 Solving the Basic Model

Since the expected progress in the innovation process as well as the costs of investment depend linearly on  $I$ , the problem will have a "bang-bang" solution. At any point in time the optimal R&D investment will be either 0 or  $R$ . As a result, the optimal investment rule reduces to a critical cutoff value for the expected total of R&D investments that still needs to be undertaken to complete the project,  $K^*$ , such that when  $K \leq K^*$ , the



firm invests at the maximum rate  $R$ , and there is no R&D investment otherwise.

As mentioned in the previous section, progress made in obtaining the innovation is very unlikely to depend on the state of the overall economy. This implies that it is not possible to replicate movements in  $dw$  with some other asset or dynamic portfolio of assets. Therefore we solve the R&D investment problem by using dynamic programming instead of contingent claims analysis as was done in, e.g., Pindyck (1993).

To apply dynamic programming we start by writing down the Bellman equation (see, e.g., Dixit and Pindyck (1994)):

$$rF(K) = \max_I \left\{ -I + \frac{1}{dt} E[dF] \right\}. \quad (6)$$

Due to Ito's lemma we obtain that

$$dF = -IF'dt + \frac{1}{2}\gamma^2 K^{2\delta} IKF''dt + \gamma K^\delta I^{\frac{1}{2}} K^{\frac{1}{2}} F'dw. \quad (7)$$

Substitution of (7) into (6) leads to the conclusion that  $F(K)$  must satisfy the following differential equation:

$$rF = \max_I \left\{ -I - IF' + \frac{1}{2}\gamma^2 K^{2\delta} IKF'' \right\}. \quad (8)$$

Because equation (8) is linear in  $I$ , the optimal R&D investment that maximizes  $F(K)$  is always either zero or the maximum rate  $R$ :

$$I = \begin{cases} R & \text{for } -1 - F' + \frac{1}{2}\gamma^2 K^{2\delta} KF'' \geq 0 \\ 0 & \text{otherwise.} \end{cases} \quad (9)$$

Equation (8) therefore has a free boundary at a point  $K^*$ , such that  $I(t) = R$  when  $K \leq K^*$  and  $I(t) = 0$  otherwise. The value of  $K^*$  is found along with  $F(K)$  by solving (8) subject to the following boundary conditions.

First, the payoff after completion is  $P$ , so we have

$$F(0) = P. \quad (10)$$

Second, it holds that when  $K$  is very large, it is not profitable to begin the project:

$$\lim_{K \rightarrow \infty} F(K) = 0. \quad (11)$$

As we have seen it is optimal to have zero investment for  $K > K^*$ . From (8) we obtain that then the value of the project  $F$  is zero as well. This can be explained by the fact that no new information about the project will be generated as long as there is no R&D investment. So, if the firm at one time concludes that the project is too expensive for performing R&D investments, this conclusion will also be drawn in the future because no investment implies that  $K$  remains constant (cf. (3)), which in turn implies that  $K$  remains larger than  $K^*$ . Therefore, the firm forever refrains from investing so that the project equals zero. Now, the value matching condition, which requires that  $F$  should be continuous in  $K$ , becomes

$$F(K^*) = 0. \quad (12)$$

Since  $F(K) = 0$  for  $K > K^*$ , we know that  $F'(K)$  equals zero as well for  $K > K^*$ . Therefore, the so-called smooth pasting condition (Dixit and Pindyck (1994)) equals

$$F'(K^*) = 0. \quad (13)$$

When  $K < K^*$  and  $I = R$ , equation (8) is a second order ordinary differential equation that must be solved numerically together with the boundary conditions (10), (11), (12) and (13). This is done for various parameter values in the remainder of this section.

We have seen before that the firm invests maximally in R&D as long as the expected total of R&D investments needed to complete the project, which is denoted by  $K$ , is below  $K^*$ , and invests nothing otherwise. We conclude that the optimal R&D investment policy is completely fixed by  $K^*$ . Hence, in order to find out in what way different parameter values affect R&D investments, it is sufficient to determine the relation between  $K^*$  and these parameters.

First, we study the effect of the overall uncertainty level on R&D investment. From equation (3) we obtain that the parameter  $\gamma$  determines the level of uncertainty; uncertainty is particularly significant when  $\gamma$  is large. Hence to find the effect of uncertainty on R&D investments we must plot  $K^*$  against  $\gamma$ , which is done in Figure 1 for the parameter values  $r = 0.05$ ,  $R = 2$  and  $P = 10$ . We see that  $K^*$  has the lowest value when uncertainty is absent. This value of  $K^*$  is obtained by applying the classical NPV rule. When uncertainty comes in  $K^*$  increases so that for each  $\gamma > 0$  there exists an interval of  $K$ -values where the classical NPV rule tells the firm to do nothing while it is in fact optimal to invest maximally. The NPV rule also fails in other stochastic investment models, and this forms the key message of a stream of recent contributions in which an analogy with the theory of options in financial markets is exploited in order to provide a much richer dynamic framework than was possible within the traditional theory of investment. An important exponent of this literature is Dixit and Pindyck (1994). In these models investment decreases with uncertainty, because the presence of uncertainty makes it valuable to wait for more information about, e.g., the price of output, before committing yourself to an increased capital stock level.

[Place Figure 1 about here].

We conclude that usually investment decreases with uncertainty, while from Figure 1 we infer that in our model it is the other way round. The reason lies in the type of uncertainty, which is technical uncertainty here rather than uncertainty about prices and regulations. Therefore, in the first place there is no value of waiting because new information about the project arrives only when the firm invests. Second, if uncertainty is large then there is a lot of uncertainty to be resolved which can be done by investing. In other words, when R&D investment is undertaken the firm learns about how many materials, manpower, etc., are required to complete the project, and the more uncertain the firm is about these matters, the more there is to learn. Obtainment of this information gives R&D investment an extra value beyond its contribution to completion of the project. In absence of uncertainty this extra value is zero because everything is already known. But the larger the uncertainty is, the larger the extra value will be, and, hence, the more the firm will invest in R&D.

As explained in the previous section, for R&D problems in particular it holds that technical uncertainty is especially large in the starting phase of the project, thus when  $K$  is

large. We modelled this by adding a term  $K^\delta$  in the uncertain part of the state equation for  $K$  (cf. (3)). Hence, the larger is  $\delta$ , the larger is the technical uncertainty during early phases of the project. In Figure 2 it is shown in what way  $K^*$  depends on  $\delta$ .

[Place Figure 2 about here].

We conclude that R&D investments increase with  $\delta$ . This is for the same reason as why investment increased with  $\gamma$ : if there is a lot of uncertainty, then much uncertainty disappears when R&D investment is undertaken which gives a large extra value to R&D investment. From Figure 2 we obtain that the parameter  $\delta$  has an enormous effect on R&D investments: this effect is much larger than the effect of the overall uncertainty parameter  $\gamma$ . Hence, ignoring  $\delta$  leads to a large underestimation of  $K^*$ , which in turn would lead to too few R&D investments.

The implication of this is that it is usually optimal to spend some resources on exploring opportunities that are completely new (thus having a large technical uncertainty) to the firm. Based on the conclusions of this first exploration phase the firm can decide to continue or abandon the project. In this way the firm can detect those innovation projects that at first sight do not look as profitable as they really are. It can be much easier to complete an innovation than one would think beforehand. If the opposite is true the firm can just stop investing before losing too much money. Notice that  $\delta$  being large implies that a lot of uncertainty is resolved during a first exploration phase. This is in contrast with the situation where  $\gamma$  is large, because then uncertainty remains large during the successive phases of the project.

Figure 3 shows how  $K^*$  depends on the discount rate for the case where uncertainty is absent ( $\gamma = 0$ ) and  $\gamma = 1$ . As we have just seen, R&D investment rises with increasing uncertainty. This is confirmed in Figure 3 where  $K^*$  increases with  $\gamma$  for a given value of  $r$ .

[Place Figure 3 about here].

Looking at the two cases in Figure 3 we see that R&D investment is decreasing with the discount rate. In the certainty case R&D investment takes place according to the NPV rule, which implies that the firm invests maximally when the net present value of the project, which is given by

$$Pe^{-rK/R} - \int_0^{K/R} Re^{-rt} dt, \quad (14)$$

is non-negative<sup>2</sup>. The project's net present value falls when the discount rate rises and this means that also R&D investment will fall, as is reflected in Figure 3. From this figure we can also conclude that this NPV effect carries over to the uncertainty case.

Summarizing the above, we conclude that an increasing discount rate has a negative effect on R&D investment, because the project's net present value decreases with  $r$ .

In general an R&D project is entirely self financed (Guerard, Bean and Andrews (1987)). Reasons for this are that, first, outsiders are reluctant to invest money in a project from which the proceeds are very uncertain. Hence, to be able to borrow the firm must pay a high interest rate to compensate the uncertainty, and this makes borrowing an unattractive means of finance. Second, the firm does not want to reveal information about an R&D project to the outside world which must be done in case of external financing (Kamien and Schwartz (1982)). Therefore, it is likely that the availability of funds for an R&D project is limited by the profit flow obtained from other operations within the firm. Figure 4 shows in what way this financing limit influences the firm's R&D investment behavior in the certainty case as well as when uncertainty is present.

[Place Figure 4 about here].

When uncertainty is absent and  $K$  is below  $K^*$ , the completion time of the project is  $K/R$ . This implies that when  $R$  is very large it is possible for the firm to complete the innovation project almost immediately. Consequently, the optimal investment decision is to invest when  $K$  falls below the project's payoff  $P$ , and do nothing when  $K > P$ . In Figure 4 this is confirmed by the fact that for  $\gamma = 0$   $K^*$  approaches an horizontal asymptote, situated at the level  $P = 10$ , from below as  $R$  increases.

When  $R$  is low the completion time is large and very sensitive to the level of  $R$ . This implies that the project's net present value (see equation (14)) depends very heavily on  $R$  too, so that the firm's optimal investment behavior changes very much with  $R$ . In Figure 4 we see that a small increase of  $R$  leads to a large increase of  $K^*$  as long as

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<sup>2</sup>Notice that the project's completion time is known when uncertainty is absent. It is  $T = K/R$ .

the completion time  $K^*/R$  is more than approximately  $4\frac{1}{2}$  years. We conclude that the presence of a financing limit is particularly sensitive to the level of R&D investment when there is not much funds available.

Figure 4 shows that also in the uncertainty case it holds that the critical level  $K^*$  is particularly sensitive to the level of the financing limit when  $R$  is small. The difference with the certainty case is that  $K^*$  is much higher, especially when  $R$  is large. The reason is that a lot of uncertainty is resolved when  $I$  is large (remember that  $I = R$  for  $K < K^*$ ). This gives a large extra value for R&D investment.

## 4 Model extensions

In this section we extend the basic model so that we can analyze the effects that three characteristics, which are typical for an innovation project, have on optimal R&D investments of the firm. We start out with including investment grants in the model, and proceed with studying the effects of spillover benefits and a decreasing payoff, respectively.

### 4.1 R&D depending on the level of the investment grant rate.

Some governments have introduced investment grants as a means to stimulate R&D activities by the firm. To see whether this finds confirmation in our model, it must turn out that for some levels of  $K$  the firm will not invest in R&D without obtaining an investment grant, while it will do so after investment grants are introduced. In other words, it must be the case that  $K^*$  increases due to the introduction of an investment grant.

If we denote the investment grant rate by  $g$  ( $0 < g < 1$ ), the value of the innovation opportunity, which in the basic model is given by equation (5), changes into

$$F(K(t)) = \max_{I(t)} E_0 \left[ P e^{-r\tilde{T}} - \int_0^{\tilde{T}} (1 - g) I(t) e^{-rt} dt \right]. \quad (15)$$

In this expression it is reflected that if the R&D investment at time  $t$  equals  $I(t)$ , the government pays  $gI(t)$  to the firm so that the net R&D expenses of the firm equal  $(1 - g)I(t)$ . The financing restriction (4) implies that the maximum R&D investment at

time  $t$  equals  $R/(1 - g)$ .

First, let us consider the deterministic case. As before a level of expected total of R&D investments to be undertaken before completion  $K^*$  can be defined, above which it is optimal not to invest and below which the firm invests maximally. This implies that for  $K < K^*$ ,  $K$  reduces with  $R/(1 - g)$  per unit of time so that the completion time equals  $(1 - g)K/R$ . If we substitute this into (15) we can derive that

$$F(K(t)) = e^{-rK(t)(1-g)/R} \left[ P + \frac{R}{r} \right] - \frac{R}{r}. \quad (16)$$

The critical level  $K^*$  can be determined by equating the marginal benefit of R&D investment to its marginal cost:

$$-F'(K^*) = 1 - g. \quad (17)$$

Equations (16) and (17) imply that  $K^*$  must satisfy

$$(1 - g)K^* = \frac{R}{r} \ln \left( \frac{Pr}{R} + 1 \right). \quad (18)$$

We conclude that  $(1 - g)K^*$  is constant so that  $K^*$  is an increasing function of  $g$ . This function is drawn in Figure 5, in which it is denoted by  $\gamma = 0$  (the certainty case). Note that  $(1 - g)K^*$  exactly equals the part of the accumulated R&D investment that must be paid by the firm.

[Place Figure 5 about here].

In Figure 5 we also find  $K^*$  as a function of  $g$  in case uncertainty is present, where  $\gamma = 1.0$ . It turns out that on this curve  $(1 - g)K^*$  is increasing, which can be concluded from the figure due to the fact that the curve for  $\gamma = 1$  lies above the dotted line on which  $(1 - g)(K^*|_{g=0, \gamma=1})$  is constant. The reason comes from the effect that  $g$  has on the variance of  $dK$  when the firm invests maximally in R&D. Then  $I = R/(1 - g)$  and from eqn. (3) we obtain that for  $I = R/(1 - g)$  it holds that

$$\text{var}(dK) = \gamma^2 K^{2\delta+1} R/(1 - g). \quad (19)$$

We conclude that  $\text{var}(dK)$  increases with  $g$ , which is due to the fact that  $I = R/(1 - g)$  increases with  $g$ , and the larger the R&D investment is, the more uncertain the firm is about the outcome. If the level of uncertainty is large, then much uncertainty disappears when R&D investment is undertaken, because, as already argued in Section 3, by investing the firm learns about how much time and effort are needed to complete the innovation project. This learning effect gives R&D investment an extra value beyond its contribution to completion of the project and this extra value increases with the level of uncertainty. This increased extra value is illustrated in Figure 5 where the distance between the  $\gamma = 1$ -curve and the dotted line increases with  $g$ , and thus with the variance of  $dK$ .

We conclude that under uncertainty an R&D investment grant has two positive effects on R&D investment. The first effect is the obvious **cost effect**, i.e. the government pays part of the investment. The second effect is the **learning effect**, i.e. donation of an investment grant gives the firm the opportunity to increase R&D investments and the more a firm can invest in R&D, the faster it can learn about the innovation project.

Of course, on the basis of this analysis we cannot conclude whether it pays for the government to distribute such a grant to the firm. To answer this question requires another model with the government as decision maker. Here we just focus on the optimal response of a firm to an exogenously given government policy.

## 4.2 R&D depending on spillover benefits.

Spillover benefits are present when it is possible for the firm to take advantage of knowledge generated outside the firm by competitors or universities. In the case of R&D firms will have difficulty preventing their competitors from taking advantage of the more general forms of scientific and engineering knowledge that are generated in the course of developing some specific product or process. But, according to Grossman and Helpman (1990) it is often necessary for the firm to invest resources in order to capture these spillover benefits from its competitors and/or from governmental institutions like universities. Therefore, below we assume that the firm can only take advantage of spillover benefits at moments that it invests in R&D.



Of course, knowledge created outside the firm increases over time. If we denote the rate of technological development outside the firm by  $w$  ( $w > 0$  and constant) and refrain from any uncertainties, the state equation for  $K$  becomes

$$dK(t) = -I(t)e^{wt}dt. \quad (20)$$

Hence, a given research outlay generates more cost reduction if this investment takes place at a later point of time, because then it benefits from the technological development outside the firm up to this later point of time.

It is clear that the completion time of the R&D project becomes time dependent. Therefore, the value of the innovation opportunity equals

$$F(K(t), t) = \max_{I(t)} \left[ P e^{-r(T(t)-t)} - \int_t^{T(t)} I(s) e^{-r(s-t)} ds \right]. \quad (21)$$

The problem is to solve (21) subject to expressions (4), (20) and  $K(T(t)) = 0$ . Like before the firm invests maximally in R&D if  $K$  is lower than a critical level, but this level is now time dependent. We denote this critical level by  $K^*(t)$ . Hence, for  $K < K^*(t)$  the expected total of R&D investments still to be undertaken to complete the project decreases with  $R e^{wt}$  at time  $t$ , so that we obtain from (20) and  $K(T(t)) = 0$ :

$$\int_t^{T(t)} R e^{ws} ds = K(t). \quad (22)$$

From (22) we derive the following expression for the time that the project is completed:

$$T(t) = \frac{1}{w} \ln \left( \frac{wK(t)}{R} + e^{wt} \right). \quad (23)$$

After substitution of (23) into (21) we get that

$$F(K(t), t) = e^{rt} \left[ \left( P + \frac{R}{r} \right) \left( \frac{wK(t)}{R} + e^{wt} \right)^{-\frac{r}{w}} - \frac{R}{r} e^{-rt} \right]. \quad (24)$$

$K^*$  can be determined by equating marginal benefits of R&D investment to its marginal cost:

$$-1 - e^{wt} F_K(K^*(t), t) = 0. \quad (25)$$

Equations (24) and (25) lead to the following expression for  $K^*(t)$ :

$$K^*(t) = \frac{R}{w} e^{wt} \left[ \left( \frac{rP}{R} + 1 \right)^{\frac{w}{r+w}} - 1 \right]. \quad (26)$$

From equation (26) we conclude that  $K^*(t)$  grows exponentially over time. Hence, if at present R&D investment is not optimal because  $K(t)$  is higher than  $K^*(t)$ , it is still possible that investing becomes optimal later because of this exponential increase of  $K^*(t)$ . The intuition behind this is clear: by postponing the R&D investment to a later date the investment generates more knowledge because then it takes advantage of the increased technological development over time.

In order to study the effect of  $w$  on R&D investment, let us concentrate on the investment decision at a particular time, say at  $t = 0$ . It is optimal to invest maximally in R&D whenever  $K$  falls below  $K^*(0)$ . To see in what way R&D investments depend on the rate of technological development  $w$ , we differentiate  $K^*(0)$  w.r.t.  $w$ :

$$\frac{dK^*(0)}{dw} = -\frac{R}{w^2} \left[ \left( \frac{rP}{R} + 1 \right)^{\frac{w}{r+w}} - 1 \right] + \frac{Rr}{w(r+w)^2} \left( \frac{rP}{R} + 1 \right)^{\frac{w}{r+w}} \ln \left( \frac{rP}{R} + 1 \right). \quad (27)$$

At first sight the sign of  $dK^*(0)/dw$  is not clear, since the first term is negative and the second term is positive. The second term is particularly important if  $P$  is large.

In Figure 6 it is depicted how  $K^*(0)$  depends on  $w$ , where the parameter values are  $P = 10$ ,  $R = 2$ ,  $r = 0.05$ , and, since we study the certainty case,  $\gamma = 0$ . We conclude that here R&D investments decrease with  $w$ . This can be explained by the fact that knowledge creation due to a given research outlay increases over time with rate  $w$ . Therefore, if  $w$  is large there is a strong incentive to postpone R&D investments, and this of course negatively affects current investments. This result confirms the well known motivation problem of R&D investments caused by spillover benefits: an innovation created by one firm provides usable information to other firms at little or no costs. While all firms stand prepared to use such information, no one firm is willing to make the investment necessary to produce it. A way to overcome this problem is granting patents

for innovations, but this prevents the diffusion of innovations which is undesirable from a welfare point of view (Tirole (1988)).

[Place Figure 6 about here].

Apparently, under the parameter values of  $P, R$  and  $r$  that we used throughout the whole paper and on which Figure 6 is based, the first term of equation (27) dominates. The second term becomes more influential for higher values of  $P$  and it is interesting to study for what values of  $P, K^*(0)$  increases with  $w$ . From Figure 7 we obtain that this for instance happens when  $P = 1000$  and  $w$  is sufficiently low. The economic intuition behind this is that when the payoff  $P$  is very large, the firm is very eager to obtain the innovation. If  $w$  increases the firm finds a new perspective in the sense that it becomes possible to obtain the innovation within a reasonable amount of time, despite of the fact that  $K$  may be large. Therefore, R&D investments increase with  $w$ .

[Place Figure 7 about here].

For most sectors the rate of technological development  $w$  will lie somewhere between 0 and 0.1. The above figures show that  $K^*(0)$  is very sensitive to  $w$  when  $w$  is low. We conclude that the rate of technological development plays an important role in determining optimal R&D investment.

### 4.3 R&D when the payoff decreases over time

In this subsection we determine the optimal R&D investment policy of the firm in case the payoff that accrues to the firm after the innovation project is completed, decreases over time. This assumption is realistic for the following two reasons.

First, the sooner the firm completes the project the longer it captures monopoly rents. Second, the number of competitors, that already has achieved the breakthrough in the innovation process, will increase over time, implying that competition in the market of the innovated product also increases over time.

As in the previous subsection we simplify the analysis by refraining from any uncertainties. Let us assume that the payoff declines at the rate of  $v$  ( $v > 0$  and constant) over time, so that the firm receives  $Pe^{-vt}$  if the project is completed at time  $t$ . This gradual

decrease of the payoff differs from Reinganum (1982), where it was assumed that the innovation payoff is constant up to a particular point of time after which it is zero. Our assumption captures the above reasons for a decreasing payoff in a better way.

The value of the innovation opportunity becomes

$$F(K(t), t) = \max_{I(t)} \left[ P e^{-vt} e^{-r(T(t)-t)} - \int_t^{T(t)} I(s) e^{-r(s-t)} ds \right]. \quad (28)$$

The problem is to solve (28) subject to expressions (1) (with  $\gamma = 0$ ), (2), and  $K(T(t)) = 0$ .

Since the payoff is a function of time, the same holds for the critical level  $K^*(t)$ . If  $K(t) < K^*(t)$  the project will be completed by time

$$T(t) = t + K/R. \quad (29)$$

Substitution of (29) into (28) eventually leads to

$$F(K(t), t) = e^{-rK(t)/R} \left( P e^{-vt} + R/r \right) - R/r. \quad (30)$$

Again we obtain  $K^*$  by equating marginal benefits of R&D investment ( $-F_K$ ) to marginal expenses (1):

$$-1 - F_K(K^*(t), t) = 0. \quad (31)$$

The critical level  $K^*(t)$  can be derived from (30) and (31):

$$K^*(t) = \frac{R}{r} \ln \left( \frac{rP}{R} e^{-vt} + 1 \right). \quad (32)$$

$K^*(0)$  has the same value as the critical level in the basic model. To see how  $K^*(t)$  develops over time let us differentiate (32) twice w.r.t. time:

$$\frac{dK^*(t)}{dt} = -vP e^{-vt} / \left\{ \frac{rP}{R} e^{-vt} + 1 \right\} < 0, \quad (33)$$

$$\frac{d^2 K^*(t)}{dt^2} = v^2 P e^{-vt} \left/ \left\{ \frac{rP}{R} e^{-vt} + 1 \right\}^2 \right. > 0. \quad (34)$$

Hence,  $K^*(t)$  decreases over time in a convex way. The reduction of the payoff over time makes it less tempting to perform R&D investments as time passes.

In order to be sure that also here it is optimal that the firm invests maximally in case  $K(t) < K^*(t)$ , it needs to be true that once  $K(t)$  is below  $K^*(t)$ , it remains below  $K^*(t)$ . Otherwise, the R&D investment at time  $t$  is wasted down the drain, because the innovation project will never be completed. This is because at the moment that  $K > K^*$  the firm stops investment, which implies that  $K$  remains constant (remember that  $\gamma = 0$ ). Since  $K^*$  is decreasing the inequality  $K > K^*$  continues to hold so that the firm will never resume investment. The following proposition shows under what parameter values the above described investment rule is still optimal. Here it is important to notice that the firm will never invest in the innovation project if  $K(0) > K^*(0)$ .

Proposition 1.

Consider the investment rule  $I(t) = R$  when  $K(t) \leq K^*(t)$ , and  $I(t) = 0$  otherwise. When this rule is applied and  $K(0) \leq K^*(0)$ , then it holds for every  $t > 0$  that  $K(t)$  remains below  $K^*(t)$  when the parameter values satisfy the following inequality:

$$\frac{R}{P} + r > v. \quad (35)$$

Proof

See Appendix.

From (33) we further obtain that

$$\frac{d(dK^*(t)/dt)}{dv} \left\{ \begin{array}{c} > \\ < \end{array} \right\} 0 \text{ for } t \left\{ \begin{array}{c} > \\ < \end{array} \right\} t^*, \quad (36)$$

where  $t^*$  is such that

$$vt^* e^{vt^*} = \frac{rP}{R}. \quad (37)$$

We conclude that a heavy payoff reduction, i.e.  $v$  large, leads to a large decrease of  $K^*(t)$  on an initial time interval.

## 5 Concluding remarks

In this paper we studied optimal R&D investment behavior of a single firm. An R&D project takes time to complete and the amount of effort, materials and time needed to obtain the innovation is in general uncertain. It may in fact not even be possible to realize the project at all. Because of this uncertainty, the fact that most assets are intangible, and that the firm is reluctant to reveal any project information to outsiders, R&D projects are usually financed by retained earnings of the firm. These features are all captured in our model.

The model is based on Pindyck (1993), in which sequential investment programs under cost uncertainty were studied. We changed some of the features of Pindyck's model so that it became more suitable for studying R&D investments. For R&D projects it holds that technical uncertainty is far more important than input cost uncertainty, and this technical uncertainty is especially significant when the project is far from being completed. Therefore, compared to Pindyck (1993), we left out input cost uncertainty, and we made technical uncertainty larger during earlier phases of the project instead of uncertainty being the same for all phases as was the case in Pindyck (1993). Inclusion of this last feature turned out to have a large positive impact on R&D investments.

Like Pindyck (1993), Kanninen (1993), and Grossman and Shapiro (1986), we also found that R&D investment increases with uncertainty. The reason is that by investing the firm gets more information about the actual total of investments that needs to be undertaken to finish the project. Generating this extra information gives an extra value to investing. Notice that in the certainty case everything is known beforehand so that then the extra value is zero, because there is no extra information to be gained.

In Kanninen (1993) the R&D process generates an immediate return in the form of increased productivity of the capital stock. This is contrary to our approach, where it takes time to complete an R&D project and a payoff is only obtained after completion. In Grossmann and Shapiro (1986) R&D investments have decreasing returns and there is no financing limit, while in our model investments have constant returns and the

investment rate cannot exceed the upperbound that comes from the fact that R&D is financed by retained earnings. Consequently, in Grossmann and Shapiro (1986) the rate of investment differs over time, while in our solution the firm either invests maximally in R&D or invests nothing at all.

Furthermore, we studied the effects of investment grants, spillover benefits, and a decreasing payoff on R&D investments. Besides the obvious decreasing cost effect, a second positive effect of investment grants is caused by additional learning. The presence of investment grants makes it possible for the firm to increase its investment rate, and the higher this investment rate, the faster it can learn about the R&D project. Spillover benefits give the firm an incentive to postpone R&D investments, because when the firm invests later it generates a higher return due to the overall technological development. However, in case of a very large payoff and spillover benefits it can be optimal to undertake R&D investments, while without spillover benefits it would be optimal to invest nothing at all. This is because spillover benefits give the firm the opportunity to complete a project without needing to use too many resources. As time passes, more other firms will obtain the innovation that the firm is working on. Therefore, the payoff of an R&D project will in general decrease over time. This causes the firm to decrease R&D investments as time passes.

An interesting topic for future research would be to relax the assumption that the profit flow obtained from other operations within the firm is constant. Then we could analyze whether it would pay for the firm to delay investment in R&D for an initial interval of time in order to accumulate retained profits and then to invest more aggressively. It seems that in real world firms do in fact sometimes resort to such an investment strategy.

## Appendix

In this appendix the proof of Proposition 1 is given.

First, notice that if it holds that  $K(t) < K^*(t) \ \forall \ t > 0$  when  $K(0) = K^*(0)$ , then this will also hold for  $K(0) < K^*(0)$ . So, let us depart from the situation that

$$K(0) = K^*(0). \quad (38)$$

Substitution of (32) into (38) gives

$$K(0) = \frac{R}{r} \ln \left\{ \frac{rP}{R} + 1 \right\}. \quad (39)$$

$K(t) < K^*(t) \ \forall \ t > 0$  implies that  $I(t) = R$ . If we substitute this into (3) (with  $\gamma = 0$ ) we obtain by using (39):

$$dK = -Rdt \Rightarrow K(t) = -Rt + K(0) = -Rt + \frac{R}{r} \ln \left\{ \frac{rP}{R} + 1 \right\}. \quad (40)$$

From (32) and (40) we can derive that  $K(t) < K^*(t)$  is equivalent with

$$-rt + \ln \left\{ \frac{rP}{R} + 1 \right\} - \ln \left\{ \frac{rP}{R} e^{-vt} + 1 \right\} < 0. \quad (41)$$

If we denote the left-hand side of (41) by  $f(t)$ , we can rewrite (38) as

$$f(0) = 0. \quad (42)$$

From (41) we obtain that

$$f'(t) = -r + v/(1 + Re^{vt}/rP). \quad (43)$$

Satisfaction of (41) requires that  $f'(0) < 0$  which is the case under expression (35). If (35) holds, we get from (43) that  $f'(t) < 0 \ \forall \ t > 0$  as well. And this in turn implies that (41) holds  $\forall \ t > 0$ . Q.e.d.



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Figure options

Figure 1:

The critical level  $K^*$  as function of  $\gamma$  under the parameter values  $\delta = 0.1, r = 0.05, R = 2$  and  $P = 10$ .

Figure 2:

The critical level  $K^*$  as function of  $\delta$  under the parameter values  $\gamma = 1, r = 0.05, R = 2$  and  $P = 10$ .

Figure 3:

The critical level  $K^*$  as function of the discount rate  $r$  for  $\gamma = 0$  and  $\gamma = 1$ , where  $\delta = 0.1, R = 2$  and  $P = 10$ .

Figure 4:

The critical level  $K^*$  as function of the upper financing limit  $R$  for  $\gamma = 0$  and  $\gamma = 1$ , where  $\delta = 0.1, r = 0.05$  and  $P = 10$ .

Figure 5:

The critical level  $K^*$  as function of the investment grant rate  $g$  for  $\gamma = 0$  and  $\gamma = 1$ , where  $\delta = 0.1, R = 2, r = 0.05$  and  $P = 10$ . The dotted line is drawn such that  $(1 - g)(K^*|_{g=0, \gamma=1})$  is constant.

Figure 6:

The critical level  $K^*(0)$  as function of the rate of technological development outside the firm  $w$ , where  $P = 10, R = 2, r = 0.05$  and  $\gamma = 0$ .

Figure 7:

The critical level  $K^*(0)$  as function of the rate of technological development outside the firm  $w$  for  $P = 100$  and  $P = 1000$ , where  $R = 2, r = 0.05$  and  $\gamma = 0$ .